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PROPAGATION OF ACOUSTIC WAVES IN A MEDIUM LOCATED IN A GRAVITATIONAL FIELD

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Certain problems of acoustic wave propagation in a medium located in a gravitational field are considered on the basis of exact solution for one-dimensional motion of the medium.

1. Fundamental equations and the general solution for one-dimensional motion of medium. Equations of one-dimensional motions of a medium in a gravitational field are of the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=-g, \quad \frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0 \qquad (1.1)$$

This system uniquely defines velocity u and density ρ for a given equation of state $p = p(\rho)$. Introducing new variables w and i, we obtain [1]

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+\frac{\partial i}{\partial x}=0, \quad \frac{\partial i}{\partial t}+u\frac{\partial i}{\partial x}+c^2\frac{\partial w}{\partial x}=0$$

$$\left(w=u+gt,\ di=\frac{dp}{\rho},\ c=\sqrt{\frac{dp}{d\rho}}\right)$$

where c is the speed of sound. After transformation of variables $t = t(w, i)$ and $x = x(w, i)$ the system of equations becomes

$$\frac{\partial x}{\partial i}-u\frac{\partial t}{\partial i}+\frac{\partial t}{\partial w}=0, \quad \frac{\partial x}{\partial w}-u\frac{dt}{dw}+c^2\frac{\partial t}{\partial i}=0 \qquad (1.2)$$

Let

$$t=\frac{\partial \psi}{\partial i}, \qquad \psi=\psi(w, i) \qquad (1.3)$$

then from (1.2) we have

$$x=wt-\frac{\partial \psi}{\partial w}-\frac{gt^2}{2} \qquad (1.4)$$

$$c^2\frac{\partial^2 \psi}{\partial i^2}+\frac{\partial \psi}{\partial i}=\frac{\partial^2 \psi}{\partial w^2} \qquad (1.5)$$

If the equation of state is given in the form $p = A\rho^n + B$, then $c^2 = i(n-1)$ and the general solution of Eq. (1.5) is

$$\psi = \frac{\partial^r}{\partial i^r}\left[F_1\left(\sqrt{2(2r+1)i}+w\right)+F_2\right)\left(\sqrt{2(2r+1)i}-w\right)\Big] \qquad (1.6)$$

$$\left(r = \frac{3-n}{2(n-1)} \quad \text{or} \quad n = \frac{2r+3}{2r+1}\right)$$

Relationships (1.3), (1.4) and (1.6) constitute the general solution of system (1.1) for $p = A\rho^n + B$, where $n = (2r+3)/(2r+1)$ and r is an integer.

In the majority of practically interesting problems it is simpler to seek a direct solution of the system of Eqs. (1.1) instead of the inverse one. Two such cases $(n = -1$ and $n = 3)$ are considered below.

1) Let $p = -A/\rho + B$. (This relationship yields Hooke's law $\sigma = \varepsilon E$, if one takes into consideration that stress $\sigma = -p$, strain $\varepsilon = \rho_0/\rho - 1$, and sets the Young's modulus $E = B$ and $E\rho_0 = A$.)

In this case it is easier to seek the solution by writing the system of equations in Lagrangian coordinates. Taking into account that $dp = -AdV$, we obtain

$$\frac{\partial u}{\partial t} - A\frac{\partial V}{\partial q} = -g, \qquad \frac{\partial u}{\partial q} - \frac{\partial V}{\partial t} = 0 \qquad (1.7)$$

$$q(x) = \int_0^x \rho_0\, dx = \int_0^{\xi(x,t)} \rho(\xi, t)\, d\xi, \qquad V = \frac{1}{\rho}, \qquad u = \left(\frac{\partial x}{\partial t}\right)_{q=\text{const}}$$

where $x(\xi, t)$ and $\xi(x, t)$ are, respectively, the Eulerian and Lagrangian coordinates of a point. The general solution of system (1.7) is of the form

$$u = F_1(q + \sqrt{A}\,t) + F_2(q - \sqrt{A}\,t) \qquad (1.8)$$

$$V = \frac{1}{\rho} = V_0 + \frac{gq}{A} + \frac{1}{\sqrt{A}}[F_1(q + \sqrt{A}\,t) - F_2(q - \sqrt{A}\,t)]$$

2) $p = A\rho^3 + B$. (This relationship approximates the equation $p = A\rho^2$.) Taking into consideration the relationship $c^2 = dp/d\rho = 3A\rho^2$ and substituting $u = {}^1/_2(\alpha + \beta)$ and $c = {}^1/_2(\alpha - \beta)$, we represent system (1.1) in the form

$$\frac{\partial \alpha}{\partial t} + \alpha\frac{\partial \alpha}{\partial x} = -g, \qquad \frac{\partial \beta}{\partial t} + \beta\frac{\partial \beta}{\partial x} = -g$$

whose general solution is

$$F_1(\alpha + gt,\ 2gx + \alpha^2) = 0, \qquad F_2(\beta + gt,\ 2gx + \beta^2) = 0 \qquad (1.9)$$

2. Let us consider some specific solutions for the cases of $p = -A/\rho + B$ and $p = A\rho^3 + B$.

1) The case $p = -A/\rho + B$. First, we consider the solution for the unperturbed medium

$$u(x = 0) \equiv 0, \qquad V(x = 0) = V_0$$

and determine the variation of density with altitude. From the general solution (1.8) we obtain

$$F_1(\sqrt{A}\,t) + F_2(-\sqrt{A}\,t) = 0, \quad F_1(\sqrt{A}\,t) - F_2(-\sqrt{A}\,t) = 0$$

taking into account that for $x = 0$ by definition $q(x) = 0$. Hence

$$F_1(q + \sqrt{A}\,t) = F_2(q - \sqrt{A}\,t) = 0, \quad V = V_0 + gq(x)/A \qquad (2.1)$$

Let us determine $q(x)$. The definition of $q(x)$ and the second of formulas (2.1) imply that

$$\frac{dq}{dx} = \left(\rho_0^{-1} + \frac{gq}{A}\right)^{-1}$$

from which

$$q(x) = \frac{\sqrt{A}}{\rho_0 g}\left(\sqrt{A + 2g\rho_0^2 x} - \sqrt{A}\right) \tag{2.2}$$

Substituting $q(x)$ into (2.1), we obtain

$$\rho = \rho_0 (1 + 2A^{-1} g\rho_0^2 x)^{-1/2}$$

Let us consider now the solution for a traveling sinusoidal wave. For a wave propagating upward the boundary conditions are

$$u_+ (x = 0) = u_0 \sin \omega t$$

and for a wave propagating downward

$$u_- (x = h) = u_0 \sin \omega t$$

From the general solution (1.8) we obtain

$$u_+ (x = 0) = F_2 (\sqrt{A}\, t) = u_0 \sin \omega t$$
$$u_- (x = h) = F_1 (q(x = h) + \sqrt{A}\, t) = u_0 \sin \omega t$$

Here and below $q(x)$ is determined by formula (2.2). From this we obtain

$$u_\pm = u_0 \sin \omega \left(t \mp \frac{q(x)}{\sqrt{A}}\right)$$
$$\rho_\pm^{-1} = \rho_0^{-1}\left(1 + \frac{2}{A}\rho_0^2 gx\right)^{1/2} \mp \frac{u_0}{\sqrt{A}} \sin \omega \left(t \mp \frac{q(x)}{\sqrt{A}}\right) \tag{2.3}$$

Here and in what follows we consider the question whether for a given equation of state $p = p(\rho)$ a difference in the velocity amplitudes of waves propagating up- and downward can exist in a medium under the same conditions. As shown by solution (2.3) in the case of the equation of state of the form $p = -A / \rho + B$, there is no difference in the velocity amplitudes. It will be shown below that for other equations of state the velocity amplitudes $u_+ (x = h)$ and $u_- (x = 0)$ are different.

2) Let us consider two specific solutions for the case of $p = A\rho^3 + B$.

Let $u \equiv 0$. In this case the general solution (1.9) must be independent of time. Hence functions F_1 and F_2 assume the form

$$F_1 (2gx + \alpha^2) = 0, \quad F_2 (2gx + \beta^2) = 0$$

from this $2gx + \alpha^2 = 2gx + \beta^2 = c_0^2$ and, consequently,

$$\alpha = -\beta = c = \sqrt{c_0^2 - 2gx}$$

For $u \neq 0$ for the wave propagating upward in the medium we have

$$F_1(\alpha_+ + gt,\ 2gx + \alpha_+^2) = 0, \qquad \beta_+ = -\sqrt{c_0^2 - 2gx} \tag{2.4}$$

Let

$$u_+(x = 0) = \begin{cases} U_+, & t_1 \leqslant t \leqslant t_2 \\ 0, & t < t_1 \quad \text{or} \quad t > t_2 \end{cases}$$

Taking into account that $u = {}^1/_2\,(\alpha + \beta)$, we obtain from this

$$\alpha_+(x=0) = 2u_+(x=0) + c_0 = \begin{cases} 2U_+ + c_0, & t + \alpha_0/g \in T_\alpha \\ c_0, & t + \alpha_0/g \notin T_\alpha \end{cases} \qquad (2.5)$$

Here and subsequently segment $T_\alpha = [t_1 + \alpha_0/g,\ t_2 + \alpha_0/g]$, $\alpha_0 = c_0$. From (2.4) and (2.5) we have

$$\alpha_+^2 + 2gx = \begin{cases} (2U_+ + c_0)^2, & t + \alpha_+/g \in T_\alpha \\ c_0^2, & t + \alpha_+/g \notin T_\alpha \end{cases}$$

From which for u_+ and c_+ we obtain

$$u_+ = \frac{1}{2}(\alpha_+ + \beta_+) = \qquad (2.6)$$

$$\begin{cases} {}^1/_2\left[\sqrt{(2U_+ + c_0)^2 - 2gx} - \sqrt{c_0^2 - 2gx}\right], & t + \alpha_+/g \in T_\alpha \\ 0, & t + \alpha_+/g \notin T_\alpha \end{cases}$$

$$c_+ = \frac{1}{2}(\alpha_+ - \beta_+) =$$

$$\begin{cases} {}^1/_2\left[\sqrt{(2U_+ + c_0)^2 - 2gx} + \sqrt{c_0^2 - 2gx}\right], & t + \alpha_+/g \in T_\alpha \\ \sqrt{c_0^2 - 2gx}, & t + \alpha_+/g \notin T_\alpha \end{cases}$$

For a wave propagating downward similar reasoning yields

$$u_- = \frac{1}{2}(\alpha_- + \beta_-) = \qquad (2.7)$$

$$\begin{cases} {}^1/_2\left[\sqrt{c_0^2 - 2gx} - \sqrt{(2U_- - \sqrt{c_0^2 - 2gh})^2 + 2g(h-x)}\right], & t + \beta_-/g \in T_\beta \\ 0, & t + \beta_-/g \notin T_\beta \end{cases}$$

$$c_- = \frac{1}{2}(\alpha_- - \beta_-) =$$

$$\begin{cases} {}^1/_2\left[\sqrt{c_0^2 - 2gx} + \sqrt{(2U_- - \sqrt{c_0^2 - 2gh})^2 + 2g(h-x)}\right], & t + \beta_-/g \in T_\beta \\ \sqrt{c_0^2 - 2gx}, & t + \beta_-/g \notin T_\beta \end{cases}$$

where

$$T_\beta = [t_1 + \beta_0/g,\quad t_2 + \beta_0/g], \qquad \beta_0 = -\sqrt{c_0^2 - 2gh}$$

Let us compare the solutions for perturbations propagating up- and downward along a two meter long aluminum rod. The perturbations are induced by pulsed load applied to the lower or upper end of the rod. We assume this load to be $p = 10$ kg/mm². For a pulsed load the relationship between velocity and voltage is $U_\pm \approx c_0 \Delta\rho_\pm / \rho \approx -c_0 \varepsilon_\pm \approx c_0 p_\pm / E$ (E is the Young modulus). Taking this relationship and the inequalities $2gh \ll c_0$ and $p/E \ll 1$ into account, from (2.6) and (2.7) we obtain

$$u_+(x=h) \approx \frac{p}{E} c_0 + \frac{pgh}{Ec_0}, \qquad u_-(x=0) \approx -\frac{p}{E} c_0 + \frac{pgh}{Ec_0}$$

$$\Delta u = |u_+(x=h)| - |u - (x=0)| \approx \frac{2pgh}{Ec_0}, \qquad \frac{\Delta u}{u} \approx \frac{2gh}{c_0^2}$$

Since for aluminum $E = 7000$ kg/mm and $c_0 = 5500$ m/sec, we have

$$\Delta u \approx 10^{-5}\ \mathrm{m/sec}, \quad u \approx 8\ \mathrm{m/sec}, \quad \Delta u / u \approx 1.25 \cdot 10^{-6}$$

3. Let us consider the linear approximation $u / c \ll 1$. The convective terms may in this case be neglected and the system of Eqs. (1.1) written as

$$\frac{\partial u}{\partial t} + \frac{nA}{n-1}\frac{\partial \rho^{n-1}}{\partial x} = -g, \quad \frac{nA}{n-1}\frac{\partial \rho^{n-1}}{\partial t} + nA\rho^{n-1}\frac{\partial u}{\partial x} = 0 \qquad (3.1)$$

$$p = A\rho^n + B$$

Let

$$\rho^{n-1}(x, t) = \rho_0^{n-1} - \frac{(n-1)\,gx}{nA} + \psi(x, t)$$

Taking into account that $c_0^2 = nA\rho_0^{n-1}$ and the term $\psi(x, t)\partial u / \partial x$ is of the second order of smallness, we transform the system of Eqs. (3.1) to the form

$$\frac{\partial u}{\partial t} + \frac{nA}{n-1}\frac{\partial \psi}{\partial x} = 0, \quad \frac{nA}{n-1}\frac{\partial \psi}{\partial t} + [c_0^2 - (n-1)\,gx]\frac{\partial u}{\partial x} = 0 \qquad (3.2)$$

Differentiating the first equation of system (3.2) with respect to t and the second one with respect to x and subtracting the second from the first, we obtain

$$\frac{\partial^2 u}{\partial t^2} = c_0^2\frac{\partial^2 u}{\partial x^2} - (n-1\ g)\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)$$

Introducing the new variable z, we obtain

$$\frac{\partial^2 u}{\partial t^2} = \left[\frac{(n-1)\,g}{c_0}\right]^2\left(\frac{\partial^2 u}{\partial z^2} + \frac{1}{z}\frac{\partial u}{\partial z}\right), \quad \frac{z^2}{4} = \left[1 - \frac{(n-1)\,gx}{c_0^2}\right]$$

Let us consider the approximate equation

$$\frac{\partial^2 u}{\partial t^2} = \left[\frac{(n-1)\,g}{c_0}\right]^2\left(\frac{\partial^2 u}{\partial z^2} + \frac{1}{z_0}\frac{\partial u}{\partial z}\right), \quad z_0 = z\,(x = 0) = 2$$

We seek a solution in the form

$$u = u_0 \exp\{j\omega t - jk_\omega (z - z_0)\}$$

Substituting into Eq. (3.3) (*), we obtain

$$k_\omega = -\frac{j}{2z_0} \pm \sqrt{\left[\frac{\omega c_0}{(n-1)\,g}\right]^2 + \frac{1}{4z_0^2}}$$

The general solution is of the form

$$u = \exp\left\{-\frac{z - z_0}{2z_0}\right\}\int_{-\infty}^{\infty}(u_1(\omega)\exp J_- + u_2(\omega)\exp J_+)\,d\omega$$

$$J_\pm = j\omega t \pm j\,(z - z_0)\sqrt{\left[\frac{\omega c_0}{(n-1)\,g}\right]^2 + \frac{1}{4z_0^2}}$$

where $u_1(\omega)$ and $u_2(\omega)$ are determined by boundary conditions. Taking into account that

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*) Editor's note. No equation of this number appears in the original Russian text.

$$z_0 = 2, \quad z = 2\sqrt{1 - \frac{(n-1)\,gx}{c_0^2}} \approx 2 - \frac{(n-1)\,gx}{c_0^2} - \left[\frac{(n-1)\,gx}{2c_0^2}\right]^2$$

and assuming that $\omega \gg (n-1)\,g\,/\,(4c_0)$, we find

$$u \approx \left[1 + \frac{(n-1)\,gx}{4c_0^2}\right] \int_{-\infty}^{\infty} (u_1(\omega)\exp J_-^\circ + u_2(\omega)\exp J_+^\circ)\,d\omega$$

$$J_\pm^\circ = j\omega t \mp j\frac{\omega x}{c_0}\left(1 + \frac{(n-1)\,gx}{4c_0^2}\right)$$

4. An experiment was carried out with the view to determining the difference in the amplitude of signals produced by waves propagating up- and downward in a medium. Short pulse signals (10-20 and 60-80 nsec) were simultaneously applied to the two two-meter long aluminum rods fixed in the same manner and insulated by brass tubes. Pulses of an amplitude of 1-2 V were supplied to piezoelectric transducers attached to rod ends and completely insulated. Signals were fed to the bottom of one rod and to the top of the other. For maximum attenuation of wave reflection the rod ends were damped by rubber. The amplitude of the output signal was of the order of 0.5-1.5 mV.

This experiment had shown that when the acoustic wave propagates upward, the amplitude of output voltage was 1.2-1.5 mV, while in the case of wave propagating downward this amplitude was 0.4-0.5 mV. The results of derived solutions were thus qualitatively confirmed. It is interesting to note that the accuracy of this experiment was sufficient for demonstrating the investigated phenomenon in spite of the small length of the rod.

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APPROXIMATE EQUATIONS FOR WAVES IN MEDIA WITH SMALL NONLINEARITY AND DISPERSION

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The method of simplifying systems of equations with small nonlinearities and dispersion is considered. Such systems differ from the linear hyperbolic system by a certain integro-differential operator with a small parameter. Method is based on the reduction of input equations to the normal form and subsequent recurrent procedure. In the case of a wave propagating along one of the characteristics of the system (single-wave processes) the first approximation by this method leads to known Burgers, Korteweg-de Vries, Klein-Gordon, and others equations which were first derived for specific physical models, and later for a more general system of